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CONJUGACY SEPARABILITY AND SEPARABLE ORBITS

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1. Introduction

Two elements (subgroups) of a group G are called *conjugacy separable* if they are conjugate in G if and only if their images are conjugate in every finite quotient of G. The whole group is termed *conjugacy separable* if each pair of its elements is conjugacy separable.

Conjugacy separable groups form a rather complicated class of groups. It is closed with respect to forming free products but not under taking subgroups, forming extensions and wreath products [8]. Restriction to the class of soluble groups does not help either. The best known result, the Theorem of Formanek [1] and Remeslennikov [7] yields that this class contains all polycyclic-by-finite groups. These groups also have all subgroups conjugacy separable (Grunewald, Segal [2]).

In this paper we will prove conjugacy separability of elements and subgroups for a class of not necessarily finitely generated nilpotent groups of finite abelian section rank. Some examples will show that there is nearly no general way to extend these results to a wider class of groups.

Regarding conjugacy as an operation of a group on itself yields another way how to extend theorems on conjugacy separability, which I found in a recent paper of Hilton and Roitberg [3].

Let Q be a group operating on a further group G. Two elements (subgroups) of G are said to be *Q-conjugate,* if there exists an element $q\in Q$ mapping the first element (subgroup) onto the second. They are termed *Q-separable* if they are Q-conjugate or if there exists a finite Q-quotient of G in which their images are not Q-conjugate. G has *separable Q-orbits* if each pair of its elements is Q-separable.

As usual Q is said to act *nilpotently* on G if it acts identically on the factors of a finite Q-invariant series of G and *almost nilpotently* if it contains a subgroup of finite index acting nilpotently on a Q-invariant subgroup of finite index of G.

Hilton and Roitberg proved orbit separability for finitely generated nilpotent groups on which a finitely generated nilpotent group acts nilpotently. We will extend

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this result and prove subgroup and orbit separability for polycyclic groups, thereby answering Question 1 of [3], and in the above mentioned class of nilpotent groups.

Before stating the precise results we need some further notation:

A group has *finite abelian section rank* if all its abelian sections (i.e. quotients of subgroups) have finite torsion-free rank and finite p-rank for every prime p. The class of soluble groups of finite abelian section rank is denoted by $\mathcal{S}_0$.

The *spectrum* $\pi(G)$ of a group G is the set of all primes p for which G has a quasicyclic p-section. $\mathcal{U}_\pi$ denotes the class of all residually finite nilpotent groups G of finite abelian section rank with spectrum π the torsion factor group $G/T(G)$ of which is π-radicable, i.e. π'-local. (So π cannot contain every prime.)

$\mathcal{U}_\pi^*$ is the class of all torsion-free $\mathcal{U}_\pi$-groups and $\mathcal{F}$ denotes the class of all finite groups.

The profinite topology, which is defined by taking the subgroups of finite index as a basis of neighbourhoods of the unit element, plays an important role in the proofs of this paper. Its properties in $\mathcal{U}_\pi$-groups have been described in [6]. Any topological term is used with respect to this topology unless it is stated otherwise.

Many arguments and ideas of this paper are based on Grunewald and Segal [2], but they must be applied carefully to $\mathcal{U}_\pi$-groups and must be improved as $\mathcal{U}_\pi$-groups may have non-closed subgroups, which cannot occur in polycyclic groups.

Notation used without explicit definition may be found in Derek Robinson's books [9].

2. Results

Theorem A. *The polycyclic-by-finite group G has separable Q-orbits if Q is a soluble-by-finite group acting on G.*

By means of counterexamples Wehrfritz [12] showed that conjugacy separability, and hence Theorem A, is not valid in nilpotent or finitely generated soluble minimax groups.

However we can prove:

Theorem B. *A $\mathcal{U}_\pi\mathcal{F}$-group G has separable Q-orbits, if Q belongs to $\mathcal{U}_\pi\mathcal{F}$ and acts almost nilpotently on G.*

We state the obvious consequence:

Theorem C. *$\mathcal{U}_\pi\mathcal{F}$-groups ue conjugacy separable.*

The set of elements conjugate to a given element in every finite quotient forms a single conjugacy class if the group is conjugacy separable. So for non conjugacy separable groups one might hope to obtain a weaker finiteness condition stating that such a set splits only into finitely many conjugacy classes. Even this is not true for torsion-free nilpotent minimax groups.

Example 1. There exists a subgroup G of the group $U(3, \mathbb{Z}[\frac{1}{2}])$ of all upper unitriangular 3×3 matrices over the ring $\mathbb{Z}[\frac{1}{2}]$ and an element $b \in G$, such that the set of elements of G, conjugate to b in every finite quotient of G, consists of infinitely many distinct conjugacy classes.

Is there a 'common' generalization of Theorems A and B to residually finite soluble groups of finite rank the Fitting subgroup of which is radicable by its spectrum? No, there is not, even not to minimax such groups:

Example 2. There exists a finitely presented torsion-free metabelian minimax group G, the Fitting subgroup of which is 2-radicable, which is not conjugacy separable. G even contains an infinite set of elements which are pairwise not conjugacy separable. G has a linear representation of degree 4 over the ring $\mathbb{Z}[\frac{1}{2}] = \{2^n m \mid n, m \in \mathbb{Z}\}$.

The group G of Example 2 is not Zariski closed in $\mathrm{GL}(4, \mathbb{Z}[\frac{1}{2}])$ and a positive answer to the following question seems possible:

Question. Let R be the integral closure of $\mathbb{Z}[1/p \mid p \in \pi]$ for a finite set π of primes in an algebraic number field and G a Zariski closed soluble subgroup of $\mathrm{GL}(n, R)$. Is G conjugacy separable?

Zariski closed sets are profinitely closed in $\mathrm{GL}(n, R)$ [13, Lemma 2]. Hence, [11, 1.21] asserts that a maximal unipotent subgroup of G is π-radicable.

The restrictions to a finitely generated ring within an algebraic number field are necessary: Wehrfritz has shown that the Zariski closed group

$$U(2, R)\lambda\langle \mathrm{diag}(1,2), \mathrm{diag}(1,x), \mathrm{diag}(1, 2-x)\rangle \quad \text{with } R = \mathbb{Z}\left[x, \frac{1}{2}, \frac{1}{x}, \frac{1}{2-x}\right]$$

and an indeterminate x is not conjugacy separable [14]. On the other hand we will establish the following example:

Example 3. Let $R = \mathbb{Z}[1/p \mid p \in \pi]$ where π contains every prime but 2. Then $U(2, R)\lambda\{\mathrm{diag}(1/s, s) \mid s \in R^*\}$ is Zariski closed in $\mathrm{GL}(2, R)$ but not conjugacy separable.

We now go back to Theorems A and B and use the well known connection between separable and profinitely closed orbits to reduce the conclusions of these theorems to statements on derivations which will also be used to prove the theorems on subgroup separability.

Let a and b be two elements of G which belong to the same Q-orbit in every finite Q-quotient of G. Then b belongs to $\bigcap_{n\in\mathbb{N}} a^Q G^n$, which coincides with the closure $\overline{a^Q}$ of a^Q. Thus we have to prove that the groups G have closed Q-orbits. Furthermore the map $d: Q \to G: q \mapsto a^q a^{-1}$ is a derivation (i.e. $d(q_1q_2)=d(q_1)^{q_2}d(q_2)$) so that Theorems A and B follow from Theorems D and E, respectively:

Theorem D. *The image of a derivation from a soluble-by-finite group into a polycyclic-by-finite group G is closed.*

For abelian G this theorem has been proved by Grunewald and Segal [2, Theorem 2].

Theorem E. *The image of a derivation from a $\mathcal{N}_\pi\mathcal{F}$-group Q into a $\mathcal{N}_\pi\mathcal{F}$-group G is closed, if Q acts almost nilpotently on G.*

Apart from orbit separability we want to get some results on subgroup separability in $\mathcal{N}_\pi\mathcal{F}$-groups. We start with an example showing that we have to replace $\mathcal{N}_\pi$ by $\bar{\mathcal{N}}_\pi$ to get a positive result:

Example 4. There exists a periodic $\mathcal{N}_\emptyset$-group of class two with non-separable subgroups.

On the other hand, by looking at finite quotients, one cannot distinguish a subgroup from its closure. Hence, we have to restrict ourselves to closed subgroups. Unfortunately, our proof needs a further restriction to almost isolated subgroups.

We term a subgroup V of a $\mathcal{N}_\pi\mathcal{F}$-group G almost isolated if its intersection $V\cap F(G)$ with the Fitting subgroup of G has finite index in its isolator

$$I_{F(G)}(V\cap F(G)) = \{g\in F(G) \mid \exists n\in\mathbb{N}\ \ g^n\in V\cap F(G)\}$$

in $F(G)$ (compare [6, 2.]).

Almost isolated subgroups of a $\mathcal{N}_\pi\mathcal{F}$-group are closed [6, Theorem A], and in a topologically finitely generated $\bar{\mathcal{N}}_\pi\mathcal{F}$-group, i.e. a $\bar{\mathcal{N}}_\pi\mathcal{F}$-group G, the Fitting subgroup $F(G)$ of which contains a finitely generated subgroup with isolator $F(G)$, every closed subgroup is almost isolated [6, 11.1]. Hence, the following two theorems are true for closed subgroups of minimax groups in $\mathcal{N}_\pi\mathcal{F}$.

Theorem F. *Let Q be a $\mathcal{N}_\pi\mathcal{F}$-group acting almost nilpotently on the $\bar{\mathcal{N}}_\pi\mathcal{F}$-group G. Then almost isolated subgroups of G are Q-separable.*

Again we state the obvious consequence on conjugacy separability:

Theorem G. *Almost isolated subgroups of $\bar{\mathcal{N}}_\pi\mathcal{F}$-groups are conjugacy separable.*

The following examples show that we must not omit the hypotheses nilpotency and radicability by the spectrum in the above theorems:

Example 5. There exists a residually finite nilpotent minimax group with infinitely many pairwise non conjugacy separable subgroups.

Example 6. There exists a finitely presented residually finite metabelian minimax group, the Fitting subgroup of which is radicable by its spectrum, which has infinitely many pairwise non conjugacy separable subgroups.

Finally we state the obvious generalization of Grunewald and Segal's theorem on conjugacy separability of subgroups of polycylic-by-finite groups:

Theorem H. *Under the action of a soluble-by-finite group Q a polycyclic-by-finite group G has Q-separable subgroups.*

Theorem D and Theorem E are proved in Sections 3 and 4, whereas the proof of Theorem F covers Sections 5–7. Section 8 contains some remarks on the proof of Theorem H. The examples are verified in Sections 9–12.

3. Proof of Theorem D

We state two lemmas concerning a derivation d from a $\mathscr{S}_0\mathscr{F}$-group Q to a $\mathscr{S}_0\mathscr{F}$-group G on which Q acts. The first lemma can be verified by simple computation.

3.1. Lemma. *Let P be a subgroup of finite index in Q and U a Q-invariant subgroup of finite index in G. Then we have:*

(a) *$d^{-1}(U)$ is a subgroup of Q.*

(b) $|Q:d^{-1}(U)|\leq|G:U|$.

(c) *d is continuous.*

(d) *if $d(P)$ is closed in G, so is $d(Q)$.*

Theorem D has been proved by Grunewald and Segal [2, Theorem 2] for abelian G. The general case follows by induction on the Hirsch number of G. The induction step in which we choose M to be a maximal abelian Q-invariant normal subgroup of G is verified by the next lemma.

3.2. Lemma. *$d(G)$ is closed in G if there exists a Q-invariant normal subgroup M of G such that*

(1) *$M^n d(G)$ is closed in G for every $n\in\mathbb{N}$.*

(2) *$d(d^{-1}(M))$ is closed in M.*

Proof. Let b belong to $d(G)$. For every $n \in \mathbb{N}$ we find elements $q_n \in Q$ and $m_n \in M^n$ such that $b = m_n d(q_n)$. Now

$$m_1 = bd(q_1)^{-1} = m_n d(q_n) d(q_1)^{-1} = m_n d(q_n q_1^{-1})^{q_1}$$

asserts

$$m_1 \in \bigcap_{n \in \mathbb{N}} M^n d(d^{-1}(M))^{q_1} = \overline{d(d^{-1}(M)^{q_1})} = d(d^{-1}(M)^{q_1}) \qquad (2).$$

Thus there exists an element $q \in d^{-1}(M) \leq Q$ satisfying $d(q)^{q_1} = m_1$ and we get

$$b = m_1 d(q_1) = d(q)^{q_1} d(q_1) = d(qq_1).$$

4. Proof of Theorem E

Following 3.1 we may assume Q and G to belong to $\mathscr{U}_\pi$ and Q to operate nilpotently on G. We prove the theorem by induction on the length of a central series of G the factors of which are transformed identically by Q.

As G is a Hausdorff group,

$$C = C_{Z(G)} Q = \bigcap_{q \in Q} \{ g \in Z(G) \mid g^q = g \}$$

is a closed subgroup of the centre $Z(G)$ of G. [6, Theorem B] asserts that C^m is closed in $Z(G)$ and in G for every natural number m. Thus G/C^m belongs to $\mathscr{U}_\pi$ [6, Proposition G] and $C^m d(G)$ is closed in G by induction.

As d is continuous (3.1) $d^{-1}(C)$ is closed in Q and hence belongs to $\mathscr{U}_\pi$ [6, Proposition G]. Now $d|_{d^{-1}(C)}$ is a homomorphism, so that $d(d^{-1}(C))$ is closed in C by [6, Proposition G]. Thus the theorem follows from Lemma 3.2.

5. Proof of Theorem F (first part)

5.1. Lemma. *Let Q act on G and P be a subgroup of finite index in Q. Two subgroups A and B of the subgroup H of finite index in G are Q-conjugate in every finite Q-quotient of G if and only if A^q and B are P-conjugate in every finite P-quotient of H for some $q \in Q$.*

Proof. (Compare [2, Corollary 1*].) Let $\{q_1 = e, q_2, \ldots, q_n\}$ be a complete set of left coset representatives of P in Q. Take any P-invariant normal subgroup N of H of finite index in H. N contains a Q-invariant normal subgroup of finite index in G. Thus there exists a number $i \in \{1, \ldots, n\}$ and an element $p_N \in P$ such that

$$A^{g_i p_N} N = BN. \qquad (1)$$

There exists an element $j \in \{1, \ldots, n\}$ such that (1) is true for every N: Otherwise for every $i \in \{1, \ldots, n\}$ we could find a subgroup N_i such that (1) is false for every $p \in P$. But for $N = \bigcap_{i=1}^n N_i$ statement (1) gives a number k and an element $p_N \in P$

such that $A^{q_k p_N} N = BN$. Now $A^{q_k p_N} N_k = A^{q_k p_N} N N_k = BNN_k = BN_k$ presents a contradiction.

Because of (1), A^{q_j} is contained in H.

5.2. We prove the theorem by induction on the Hirsch number $h(G)$ of G, which we may assume to be positive. Let U and V be two almost isolated subgroups of G. Denote by P the $\mathscr{U}_\pi$-subgroup of finite index of Q which operates nilpotently on a Q-invariant normal $\mathscr{U}_\pi$-subgroup N of finite index in G.

$C = C_{Z(N)}P$ has positive Hirsch-number and is isolated in N as $Z(N)$ is isolated in N [8, 2.25] and uniquely π-radicable. Thus N/C belongs to $\mathscr{U}_\pi$ [6, Proposition G] and G/C^m to $\mathscr{U}_\pi \cdot \mathscr{F}$ for every $m \in \mathbb{N}$. By induction we may assume UC^m and VC^m to be Q-conjugate for every $m \in \mathbb{N}$, i.e. for every $m \in \mathbb{N}$ there exists $q_m \in Q$ satisfying

$$U^{q_m} C^m = VC^m. \tag{1}$$

We may assume $q_1 = e$. $L = UC$ is closed in G, and so is $R = N_Q L$ which consequently belongs to $\mathscr{U}_\pi \cdot \mathscr{F}$ ([6, Proposition G]). Obviously R contains the elements q_n, so that U and V are R-conjugate in every finite R-image of L and we may assume $Q = R$ and $G = L$.

We now verify that

$$X = U \cap C \quad \text{and } Y = V \cap C \quad \text{are } Q\text{-conjugate.} \tag{2}$$

It follows from (1) that X and Y are Q-conjugate in every finite Q-quotient of G. Thus for some $q \in Q$ X^q and Y are P-conjugate in every finite P-quotient of G (Lemma 5.1). But P acts trivially on X^q and Y and both subgroups are closed, so X^q equals Y.

In the remainder of the proof we distinguish two cases:

(1) $X = \langle 1 \rangle$ (split case),

(2) $X \neq \langle 1 \rangle$ (non-split case),

and finish the proof for the second case:

We may assume $X = Y$. Now $P_1 = N_Q X$ and $H = N_G X$ have finite index in Q and G respectively, so that we can find an element $q \in Q$ such that U^q and V are P_1-conjugate in every finite P_1-quotient of H (5.1). Thus U^q/X and V/X are P_1-conjugate in every finite P_1-quotient of H/X.

Now X is almost isolated in N, so that the torsion subgroup of N/X is finite and G/X belongs to $\mathscr{U}_\pi \cdot \mathscr{F}$. (This is the only part of the proof where we need U and V to be almost isolated instead of closed.) By induction there exists an element $h \in H$ such that $(U^q/X)^h = V/X$. Hence we have $U^{qh} = V$ which proves the 'non-split case'.

The split case needs some further preparations. In this case U and V are complements of C in G and we have to prove that complements which are Q-conjugate in every finite Q-quotient of G are Q-conjugate.

It is a well known fact that the complements of C in G are in 1-1 correspondence with the elements of $\mathrm{Der}(U,C)$, the abelian group of all derivations from U into C.

In Section 6 we investigate this group and finish the proof of Theorem F in Section 7.

6. Derivations

Let $G=UC$ be a Q-group with $C \trianglelefteq G$ and $U \cap C = \langle 1 \rangle$. The set of all complements of C in G is denoted by $\mathcal{K}$. For $K \in \mathcal{K}$ and $g \in G$ we write $g = g_K g_{K,C}$ for the unique elements $g_K \in K$ and $g_{K,C} \in C$. The following two lemmas are well known and easily verified by direct computation (compare [2, 1.]).

6.1. Lemma

$$f: \begin{cases} \mathrm{Der}(U,C) \to \mathcal{K} \\ d \mapsto \{ud(u) \mid u \in U\} \end{cases}$$

and

$$h: \begin{cases} \mathcal{K} \to \mathrm{Der}(U,C) \\ K \mapsto d : u \mapsto (u_{K,C})^{-1} \end{cases}$$

are bijections and inverse to each other.

6.2. Lemma. *Q acts on* $\mathrm{Der}(U,C)$ *via* $d^q(u) = d((u^{q^{-1}})_U)^q$ *and*

$$D: \begin{cases} Q \to \mathrm{Der}(U,C) \\ q \mapsto h(U^q) \end{cases}$$

is a derivation.

We want to apply Theorem E to the above derivation and verify the necessary hypothesis. More generally we state:

6.3. Lemma. *For an arbitrary set π of primes we put $R = \mathbb{Z}[1/p \mid p \in \pi]$. Let H be an $\mathcal{S}_0\mathcal{F}$-group and W a right RH-module, which is a torsion-free abelian group of finite rank. We also suppose $\pi(H) \subseteq \pi(W) = \pi$.*

(a) $\mathrm{Der}(H,W)$ *allows a natural R-module structure:* $n\mathrm{Der}(H,W) = \mathrm{Der}(H,nW)$ *for every $n \in \mathbb{N}$.*

(b) $\pi(\mathrm{Der}(H,W)) = \pi$ *if* $\mathrm{Der}(H,W) \neq \{0\}$.

(c) $\mathrm{Der}(H,W)$ *has finite rank if $H/C_H W$ is finitely generated.*

Proof. (a) For $r \in R$, $d \in \mathrm{Der}(H,W)$ the map $(rd): h \mapsto r(d(h))$ is a derivation. We obviously have $n\mathrm{Der}(H,W) \subseteq \mathrm{Der}(H,nW)$. For $d \in \mathrm{Der}(H,nW)$ we define $\tilde{d} \in \mathrm{Der}(H,W)$ by $\tilde{d}(h) = w$ if $d(h) = nw$. $\tilde{d}$ is well defined as W is torsion-free. Thus we get $n\mathrm{Der}(H,W) = \mathrm{Der}(H,nW)$.

(b) Elements of $\mathrm{Der}(H, W)$ are divisible by π-numbers and elements of W not by any π'-number (π' is the set of primes not in π).

(c) A derivation d of $\mathrm{Der}(H, W)$ restricted to $C_H W$ is a homomorphism. Hence its kernel contains $T = T(C_H W)$ and we get $\mathrm{Der}(H, W) = \mathrm{Der}(H/T, W)$. So we may assume $T = E$. The Fitting subgroup F of $C_H W$ is torsion-free nilpotent and $C_H W/F$ is finitely generated [9, 10.33]. Thus H/F is generated by $\{h_1 F, \dots, h_n F\}$ for suitable $h_i \in H$.

Using a Malcev-base $\{h_{n+1}, \dots, h_m\}$ of F we put $B = \{h_1, \dots, h_m\}$ and $A = W^B$ the set of all functions from B to W. As in (a) we may define an R-module structure on A. Obviously A is an abelian group of finite rank, so it is enough to verify that the R-homomorphism $\mathrm{Der}(H, W) \to A$: $d \mapsto d|_B$ is injective. But this follows from the fact that $d|_F$ is a homomorphism and W is torsion-free.

7. Proof of Theorem F (split case)

We use the notation of Section 5. In order to apply Theorem E to the derivation $D: Q \to \mathrm{Der}(U, C)$ we have to find a submodule of finite index in $\mathrm{Der}(U, C)$ on which P acts nilpotently.

Now $U_1 = U \cap N$ centralizes C, so that we can regard C as a U/U_1-module and we have $\mathrm{Der}(U_1, C) = \mathrm{Hom}(U_1/T, C)$ with $T/U_1' = T(U_1/U_1')$ the torsion subgroup of U_1 modulo its derived group. Furthermore by [4, VI8.1] the restriction map $j: \mathrm{Der}(U, C) \to \mathrm{Der}(U_1, C) = \mathrm{Hom}(U_1/T, C)$ has kernel

$$\ker j = \{d \in \mathrm{Der}(U, C) \mid d|_{U_1} = 0\} \cong \mathrm{Der}(U/U_1, C).$$

The subgroups $Z_i = Z_i(N) \cap U_1$ form a central series of U_1 of length s, say. Direct calculation now shows that P acts identically on the factors of the series $\{0\} = A_s \trianglelefteq \cdots \trianglelefteq A_0 = \mathrm{Hom}(U_1/T, C)$ with

$$A_i = \{\beta \in \mathrm{Hom}(U_1/T, C) \mid Z_i T/T \le \ker \beta\}.$$

Therefore we get

$$[\mathrm{Der}(U, C), {}_s P] \le \ker j = \mathrm{Der}(U/U_1, C). \tag{1}$$

$\mathrm{Der}(U/U_1, C)$ contains the submodule $\mathrm{Ider}(U/U_1, C) = \{d_c : uU_1 \mapsto c^u c^{-1}\}$ of inner derivations, which has finite index in $\mathrm{Der}(U/U_1, C)$, say r [4, VI16.5] and which is a Q-homomorphic image of C:

$$d_c^q(uU_1) = (d_c(u^{q^{-1}} U_1))^q = (c^{u^{q^{-1}}} c^{-1})^q = c^{qu}(c^q)^{-1} = d_{c^q}(uU_1).$$

Thus P acts trivially on $\mathrm{Ider}(U/U_1, C)$ and we get

$$[r\mathrm{Der}(U, C), {}_{s+1} P] = \{0\}. \tag{2}$$

By 6.3(c), $r\mathrm{Der}(U, C)$ has finite index; so we may apply Theorem E to the Derivation $D: Q \mapsto \mathrm{Der}(U, C)$ and we can finish the proof of the theorem:

For every $m\in\mathbb{N}$ we have $U^qC^m=VC^m$ for some $q\in Q$. Hence for every $\dot{u}\in U$ and $m\in\mathbb{N}$ there exist an element $c\in C$ with $\dot{u}\in U^qcC^m=VcC^m$ and elements $u\in U$; $v\in V$; $a,b\in C^m$ satisfying $\dot{u}=u^qca=vcb$. Hence

$$(h(V)-D(q))(\dot{u})=(h(V)-h(U^q))(\dot{u})=(\dot{u}_{C,V})^{-1}\dot{u}_{C,U^q}=b^{-1}c^{-1}ca\in C^m.$$

Thus $h(V)$ belongs to

$$\bigcap_{m\in\mathbb{N}} D(Q)+\mathrm{Der}(U,C^m)=\bigcap_{m\in\mathbb{N}} D(Q)+m\mathrm{Der}(U,C),$$

which equals $\overline{D(Q)}$. But $D(Q)$ is closed, so that $h(V)$ is an element of $D(Q)$, i.e. there exists an element $q\in Q$ such that $h(V)=h(U^q)$. Finally h is injective (6.1) so that we get the desired equality $U^q=V$.

8. Proof of Theorem H

Using the arguments of Sections 5–7 there is no problem to give the details of a proof of Theorem H. In the induction step one uses the last non-trivial subgroup M of the derived series of a torsion-free polycyclic subgroup of finite index in G. [2, Theorem 1*] takes care of the fact that M is not necessarily a trivial P-module for any subgroup P of finite index in Q. Instead of Theorem E one applies Theorem D or Grunewald and Segal's Theorem 2 [2] to the derivation $D: Q\mapsto \mathrm{Der}(U,M)$ in the split case.

9. Example 1 and Example 5

$$G=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}\;\middle|\; x,y\in\mathbb{Z}, z\in\mathbb{Z}[\tfrac{1}{2}]\right\}$$

is a torsion-free nilpotent minimax group of class 2 with centre

$$C=\left\{\begin{pmatrix}1&0&z\\0&1&0\\0&0&1\end{pmatrix}\;\middle|\; z\in\mathbb{Z}[\tfrac{1}{2}]\right\}.$$

We put

$$b=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad a_i=\begin{pmatrix}1&0&2^{-i}\\0&1&0\\0&0&1\end{pmatrix}\quad (i\in\mathbb{N}).$$

Because of

$$b^{-1}b^G=\left\{\begin{pmatrix}1&0&y\\0&1&0\\0&0&1\end{pmatrix}\;\middle|\; y\in\mathbb{Z}\right\}$$

the group $C/b^{-1}b^G$ is isomorphic to the Prüfer-2-group. Thus neither $b^{-1}b^G$ nor b^G are closed.

All the elements ba_i ($i\in\mathbb{N}$) belong to $\overline{b^G}$, but their conjugacy classes are different. Furthermore the subgroups $\langle ba_i\rangle$ ($i\in\mathbb{N}_0$) are pairwise non-conjugate, but are conjugate in every finite image of G.

10. Example 2 and Example 6

$$N=\left\{\begin{pmatrix}1&0&0&z&w\\0&1&0&y&0\\0&0&1&x&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\;\middle|\; x,y,z,w\in\mathbb{Z}[\tfrac12]\right\}$$

and its subgroup M obtained by putting $w=0$ are torsion-free abelian 2-radicable groups which are normalized by the commuting matrices

$$\alpha=\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

and the diagonal matrices $\beta=\operatorname{diag}(1,1,1,2,1)$ and $\gamma=\operatorname{diag}(1,1,1,1,2)$. ζ, η and ω denote the matrices in N which have every non-diagonal entry zero apart from $z=1$, $y=1$ and $w=1$, respectively.

By [5, Lemma] $M\lambda\langle\beta\rangle$ and $\langle\omega,\gamma\rangle$ are finitely presented and so are $H=M\lambda\langle\alpha,\beta\rangle$ and $G=N\lambda\langle\alpha,\beta,\gamma\rangle$.

10.1. The elements $\eta\zeta^{2^r}$ ($r\in-\mathbb{N}$) are pairwise not conjugacy separable in H:

$$(\eta\zeta^{2^r})^H=\{\eta^{\alpha^n\beta^m}\zeta^{2^r\beta^m}\mid m,n\in\mathbb{Z}\}=\{\eta^{2^m}\zeta^{(2^r-n)2^m}\mid m,n\in\mathbb{Z}\}$$

contains $\eta\zeta^{2^s}$ only if $r=s$. Furthermore, $\overline{\eta^{-1}\eta^H}$ contains $\zeta^{\mathbb{Z}}$ and hence its closure $\zeta^{\mathbb{Z}[\frac12]}$. So all the elements $\eta\zeta^{2^r}$ belong to $\overline{\eta^H}$ and are conjugate in every finite quotient of H.

10.2. The subgroups $U_r=\langle\gamma\eta\zeta^{2^r}\rangle$ ($r\in-\mathbb{N}$) are closed in G and pairwise not conjugacy separable in G:

The subgroups U_r are unipotent-free and hence closed in G [6, Theorem A]. By 10.1 all these subgroups are conjugate in every finite image of G but

$$
\begin{aligned}
U_r^G &= \{(\gamma\eta\zeta^{2^r})^{n\alpha^i\beta^j\omega^k} \mid i,j,k,n \in \mathbb{Z}\} \\
&= \{(\gamma^{\omega^k}\eta^{\alpha^i\beta^j}\zeta^{2^r\beta^j})^n \mid i,j,k,n \in \mathbb{Z}\} \\
&= \{\gamma^n\omega^{k-2^n k}\eta^{n2^j}\zeta^{n2^r2^j-i2^j} \mid i,j,k,n \in \mathbb{Z}\}
\end{aligned}
$$

contains $\gamma\eta\zeta^{2^s}$ only if $r=s$.

11. Example 3

We put $b=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and compute

$$b^G=\left\{\begin{pmatrix}1 & s^2\\ 0 & 1\end{pmatrix}\middle|\, s\in R^*\right\}.$$

By [6, Theorem A] this set is closed in G if and only if it is closed in $F(G)=U(2,R)$. Hence it is closed if and only if $(R^*)^2$ is closed in R. But 17 is a square in the ring of 2-adic numbers [10, II Théorème 4] which is equal to the profinite completion of R. Hence, $(R^*)^2$ is not closed in R.

12. Example 4

The subgroups

$$U_p=\left\{\begin{pmatrix}1&0&0\\0&1&r\\0&0&1\end{pmatrix}\middle|\, r\in\mathbb{Z}/p\mathbb{Z}\right\} \quad\text{and}\quad V_p=\left\{\begin{pmatrix}1&0&r\\0&1&r\\0&0&1\end{pmatrix}\middle|\, r\in\mathbb{Z}/p\mathbb{Z}\right\}$$

are conjugate in $U(3,\mathbb{Z}/p\mathbb{Z})$. As every normal subgroup of finite index of $G=\bigoplus_{p\text{ prime}} U(3,\mathbb{Z}/p\mathbb{Z})$ contains almost all factors $U(3,\mathbb{Z}/p\mathbb{Z})$, its subgroups

$$U=\bigoplus_{p\text{ prime}} U_p \quad\text{and}\quad V=\bigoplus_{p\text{ prime}} V_p$$

are conjugate in every finite quotient of G. But an element of G conjugating U and V would have a non-trivial component in every factor $U(3,\mathbb{Z}/p\mathbb{Z})$, which is not possible.

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